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Let n be natural number such that $2n+1$ and $3n+1$ are perfect squares. Prove that n is divisible by 40

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Let $2n+1=q^2, 3n+1=p^2$. Since q^2 odd then $q=2k+1$ for some $k\in\mathbb{Z}$ and, therefore, $2n+1=(2k+1)^2 \Leftrightarrow n=2k(k+1)$.

Hence, $p^2=3\cdot 2k(k+1)+1=6k^2+6k+1 \Rightarrow p$ is odd, that is $p=2t+1$, for some integer t. Therefore, $p^2=6k^2+6k+1 \Leftrightarrow (2t+1)^2=6k^2+6k+1 \Leftrightarrow$ $2t(t+1)=3k(k+1)$. Then $3n=6k(k+1)=4t(t+1)\,\vdots\,8 \Rightarrow n\,\vdots\,8$.

Since $p^2+q^2=5n+2$ then $p^2+q^2\equiv 2(\operatorname{mod}5) \Leftrightarrow \begin{cases} p^2\equiv 1(\operatorname{mod}5) \\ q^2\equiv 1(\operatorname{mod}5) \end{cases} \Rightarrow$

$n=p^2-q^2\equiv 0(\operatorname{mod}5)$. Thus, $n\,\vdots\,40$.